

Original articles

Convertible bond pricing with partial integro-differential equation model

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Abstract

In this paper, we introduce the concept of Exponential Variance Gamma (EVG) model to the valuation of convertible bond (CB). Rather than evaluating derivatives with standard Black–Scholes approach, we describe the dynamic underlying asset log price with VG process, which is one of classical Lévy processes with non-normal distribution but skewness and leptokurtosis. For numerical purpose, we develop a discrete scheme with stability and convergence, which combines so-called multi-stage compound-option model (MCO) and explicit–implicit difference method (EXIM) to discretize the partial integro-differential equation (PIDE). By comparing our results with Black–Scholes approach, we can show that because of the ability to capture skewness and leptokurtosis features, the new approach does provide a lower price for the valuation of CB.

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1. Introduction

Convertible bond (CB) is a contingent claim providing investors a right, in which investors can choose to convert the bond into a predetermined number of stocks with a pre-specified price, or to hold the bond till maturity to receive coupons and the principal.

It is difficult to price CB because it cannot be simply considered as a combination of equity and bonds. Its price depends on several variables such as underlying stock price, interest rate, credit risk and interaction between these components. Moreover some special types of convertible bonds with sophisticated features such as call clause require more flexible valuation models.

Theoretical research on convertible bond pricing can be divided into three approaches. First approach prices CB depending on the firm value, which is initiated by [16] and extended by [4,5], see also in [20] and [6]. The factors such

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as volatility and firm value in these methods are unobservable. The second approach assumes the stock price as the underlying state variables to price the CB. In [19], they develop a pricing model based on a finite difference method with the stock price as stochastic variable. [14] extends the work by introducing Ho–Lee interest rate model [13]. The third approach prices CB by extending the former models with credit risk variable. [7] incorporates the reduced-form [10] credit-risk model into CB pricing framework and uses a trinomial tree to find the numerical results. Similar credit-risk approaches are followed by [3,15,22].

In those papers above, they all focus on describing the firm value or underlying stock price by Geometric Brownian motion as Black–Scholes model. As is known to all, the probability distribution of empirical log returns of contingent claims presents the characteristics of skewness and leptokurtosis (high peaks and heavy tail). But the normal distribution cannot capture these features, and produces a so-called volatility skew or smile in option pricing model. By introducing extra parameters, Variance Gamma (VG) process has a number of good mathematical properties and has been proven to explain a number of economic findings. Mathematically, the distributions have nice properties such as skewness and leptokurtosis and fat tails. Economically, [17] shows that VG process is able to explain the well documented biases “volatility smile” in equity options. As a result, [8,23,25] introduced the VG process into portfolio optimization, permanent convertible bonds pricing or interest rate swap pricing respectively.

In this paper, we introduce the Exponential VG process into the American-style CB pricing framework and establish the partial integro-differential equation (PIDE) model. For numerical calculation, we incorporate the so-called multi-stage compound-option (MCO) model¹ and explicit–implicit difference method. By comparing the results with the classical Black–Scholes approach, we find that the differences of CB prices and the differences of optimal conversion prices between EVG model and GBM model are significant.

The outline of this paper is as follows. The Variance Gamma process and exponential VG model are summarized in Section 2. In Section 3, we formulate the valuation of CB with EVG model and derive the corresponding PIDE. Section 4 explains the discretization scheme for the PIDE. Numerical results with both EVG model and GBM model are presented in Section 5. Section 6 concludes with a summary and comments on further research.

2. Exponential variance gamma model

The Variance Gamma (VG) process was first introduced in financial modeling by [18] to cope with the shortcomings of Black–Scholes model. It provides a model (VG model) for log-return distribution that incorporates both the leptokurtosis and skewness features.

2.1. The variance gamma process

The Variance Gamma process $X = \{X_t^{VG}, t \geq 0\}$ with constant parameters $\sigma > 0$, $v > 0$ and $\theta \in \mathbb{R}$, introduced by [18], is a pure jump finite variation process with infinite activity. Its Lévy triplet is given by $[\gamma, 0, \nu_{VG}(dx)]$, where

$$\nu_{VG}(dx) = k(x)dx \quad (1)$$

$$k(x) = \frac{e^{-\lambda_p x}}{vx} 1_{\{x>0\}} + \frac{e^{-\lambda_n |x|}}{v|x|} 1_{\{x<0\}} \quad (2)$$

$$\lambda_p = \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2 v}} - \frac{\theta}{\sigma^2} \quad (3)$$

$$\lambda_n = \sqrt{\frac{\theta^2}{\sigma^4} + \frac{2}{\sigma^2 v}} + \frac{\theta}{\sigma^2} \quad (4)$$

$$\gamma = \frac{\lambda_p(e^{-\lambda_n} - 1) - \lambda_n(e^{-\lambda_p} - 1)}{v\lambda_n\lambda_p} \quad (5)$$

¹ MCO model was initially proposed by [21] as a natural way to approximate the value of an American option with Black–Scholes model, see also in [11]. A simple compound-option is an option on another option, and MCO is a series of sequentially ordered compound-options.

The accumulate characteristic function of VG process, which is often called the characteristic exponent, satisfies the following Lévy–Khintchine formula,

$$E[\exp(iuX_t^{VG})] = \exp(\phi(u)t) \quad (6)$$

$$\phi(u) = i\gamma u + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux1_{\{|x|<1\}})k(x)(dx). \quad (7)$$

Another way of defining a VG process is by treating it as (Gamma) time-changed Brownian motion with drift. More precisely, let $\Gamma(t; a, b)$ be a Gamma process with shape parameter $a > 0$ and scale parameter $b > 0$. Then $g_t^\nu = \Gamma(t; 1, \nu)$ is a special Gamma process with parameters $a = 1$ and $b = \nu$; so the VG process X_t^{VG} , with parameters $\sigma > 0$, $\nu > 0$ and θ , can alternatively be defined as

$$X_t^{VG} = \theta g_t^\nu + \sigma W(g_t^\nu) \quad (8)$$

with characteristic function as

$$E[\exp(iuX_t^{VG})] = \left(\frac{1}{1 - i\theta\nu u + \sigma^2 u^2 \nu / 2} \right)^{t/\nu}. \quad (9)$$

Of course, the two definition are equivalent. Similar to formula (A.5), We can deduce that :

$$\int_{-\infty}^{+\infty} x 1_{\{|x|<1\}} k(x)(dx) = \gamma.$$

And from Lemma 4.1 in [24], we have

$$\int_{-\infty}^{+\infty} (e^{iux} - 1)k(x)(dx) = \frac{1}{\nu} \ln \frac{\lambda_p \lambda_n}{(\lambda_p - iu)(\lambda_n + iu)}.$$

As a result

$$\begin{aligned} \phi(u) &= i\gamma u + \int_{-\infty}^{+\infty} (e^{iux} - 1 - iux 1_{\{|x|<1\}})k(x)(dx) \\ &= i\gamma u + \int_{-\infty}^{+\infty} (e^{iux} - 1)k(x)(dx) - i\gamma u \\ &= \frac{1}{\nu} \ln \frac{\lambda_p \lambda_n}{(\lambda_p - iu)(\lambda_n + iu)}. \end{aligned}$$

So we can easily deduce that the formula(6) and the formula(9) are equal to each other.

The VG process is a Markov process, its infinitesimal generator is an integro-differential operator defined by the following expression for $f \in C_0^2(R)$:

$$\begin{aligned} L^{VG} f(x) &= \lim_{t \rightarrow 0} \frac{E[f(x + X_t^{VG})] - f(x)}{t} \\ &= \gamma \frac{\partial f}{\partial x} + \int [f(x + y) - f(x) - y 1_{\{|y|<1\}} \frac{\partial f}{\partial x}] k(y) dy. \end{aligned} \quad (10)$$

2.2. Exponential VG model and PIDE

Let $\{S_t | t \in [0, T]\}$ be the price of a financial asset modeled as a stochastic process on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{Q})$. We assume the stock price is modeled as the exponential of a VG process (EVG model) normalized by its expectation, which is in form of (11).

$$S_t = S_0 \exp((m - q)t + X_t^{VG}) \quad (11)$$

where q is the dividend rate.

Under the hypothesis of absence of arbitrage there exists a risk-neutral measure equivalent to \mathcal{Q} under which $\{e^{-(r-q)t} S_t\}$ is a martingale. We will assume therefore without loss of generality that \mathcal{Q} is already a risk-neutral

measure. Under this measure the VG risk neutral process for the stock price is given by (12):

$$S_t = S_0 \exp((r - q + \omega)t + X_t^{VG}) \quad (12)$$

where the normalization factor $e^{\omega t}$ ensures that $E[S_t] = S_0 e^{(r-q)t}$. It follows from the characteristic function evaluated at $-i$ that

$$\omega = \frac{1}{v} \ln(1 - \theta v - \sigma^2 v/2).$$

Denote $X_t := (r - q + \omega)t + X_t^{VG}$, the Lévy triple of X_t is $[\gamma', 0, \nu_{VG}(dx)]$, where²

$$\gamma' = (r - q + \omega) + \int_{-\infty}^{\infty} x 1_{\{|x| < 1\}} k(x) dx. \quad (13)$$

Thus, the infinitesimal generator under risk-neutral measure \mathcal{Q} is,

$$L^X(f) = (r - q + \omega) \frac{\partial f}{\partial x} + \int_{-\infty}^{\infty} (f(x + y) - f(x)) k(y) dy. \quad (14)$$

The value of a contingent claim with terminal payoff $H(S_T)$ is defined as a discounted conditional expectation of its terminal payoff under risk-neutral measure \mathcal{Q} : $U_t = E[e^{-r(T-t)} H(S_T) | \mathcal{F}_t]$. From the Markov property, $U_t = U(S, t)$ where

$$U(S, t) = E[e^{-r(T-t)} H(S_T) | S_t = S]. \quad (15)$$

Following from the Feynman–Kac formula for Lévy processes, we obtain

$$\frac{\partial U}{\partial t}(S, t) + L^S U(S, t) - rU(S, t) = 0 \quad U(S, T) = H(S) \quad (16)$$

where L^S has similar form to L^X :

$$L^S U(S, t) = (r - q + \omega) S \frac{\partial U}{\partial S}(S, t) + \int_{-\infty}^{\infty} [U(Se^y, t) - U(S, t)] k(y) dy \quad (17)$$

3. Convertible bond pricing with VG model

A standard, plain-vanilla convertible bond is a bond that additionally offers the investor the option to exchange it for a predetermined number of stocks during a certain, predefined period of time. In this paper, we consider an American-style CB which can be converted into underlying stocks any time before and on the expiry date. Suppose that a CB has K discrete coupon payments within its lifetime, as long as the CB is not converted. Let $T_k (k = 1, 2, \dots, K)$ denote the preset coupon delivery dates of the CB and $R_k (k = 1, 2, \dots, K)$ the coupon rate, by which the amount of cash received by the holder of the CB at the time T_k is calculated. Denote the face value of CB as C while the preset conversion price as P , which is assumed to be constant in the lifetime of CB. The conversion ratio $I = \frac{C}{P}$ as a result.

As the price of CB is discontinuous on coupon date, we divide the lifetime of CB into K sub-intervals $[T_{k-1}, T_k] (k = 1, 2, \dots, K)$. Denote $U_k(S, t)$ the CB price on the k th interval, thus the PIDE for the CB is as follows:

$$\frac{\partial U_k}{\partial t}(S, t) + L^S U_k(S, t) - rU_k(S, t) = 0 \quad \text{for } t \in [T_{k-1}, T_k]. \quad (18)$$

By making the change of variables $x = \log(S_t/S_0)$ and $\tau = T_k - t$, we present the formulation of (18) in the logarithmic price.

Denote:

$$V_k(x, \tau) = U_k(S, t) \quad (19)$$

$$\frac{\partial V_k}{\partial x}(x, \tau) = S \frac{\partial U_k}{\partial S}(S, t) \quad (20)$$

² See Appendix A.

$$\frac{\partial V_k}{\partial \tau}(x, \tau) = -\frac{\partial U_k}{\partial t}(S, t) \quad (21)$$

$$V_k(x + y, \tau) = U_k(Se^y, t) \quad (22)$$

and the optimal logarithmic asset value at which the CB should be converted:

$$c_k(\tau) = \inf\{x \in \mathbb{R} | V_k(x, \tau) < IS_0e^x\} \quad \tau \in [0, T_k - T_{k-1}]. \quad (23)$$

As a result, formula (18) should be reformulated with $V_k(x, \tau)$ as:

$$\frac{\partial V_k}{\partial \tau} = \mathcal{L}V_k - rV_k \quad \tau \in [0, T_k - T_{k-1}], x \in (-\infty, c_k(\tau)) \quad (24)$$

with initial and boundary conditions³

$$V_k(x, 0) = H_k(x), \quad \text{for } x \in \mathbb{R} \quad (25)$$

$$V_k(-\infty, \tau) = e^{-r\tau} H_k(-\infty), \quad \text{for } \tau \in [0, T_k - T_{k-1}] \quad (26)$$

$$V_k(c_k(\tau), \tau) = IS_0 \exp(c_k(\tau)), \quad \text{for } \tau \in [0, T_k - T_{k-1}] \quad (27)$$

where

$$\mathcal{L}V_k = (r - q + \omega) \frac{\partial V_k}{\partial x} + \int_{-\infty}^{\infty} [V_k(x + y, \tau) - V_k(x, \tau)] k(y) dy,$$

$$H_k(x) = \begin{cases} \max(IS_0e^x, C(1 + R_k)) & \text{if } k = K \\ \max(IS_0e^x, V_{k+1}(x, T_{k+1} - T_k) + CR_k) & \text{if } k < K. \end{cases}$$

Two extra conditions on the solution must be imposed,⁴

$$\frac{\partial V_k}{\partial \tau} \geq \mathcal{L}V_k - rV_k, \quad \text{for } \tau \in [0, T_k - T_{k-1}], x \in (c_k(\tau), \infty) \quad (28)$$

$$V_k(x, \tau) \geq IS_0e^x, \quad \text{for } \tau \in (0, T_k - T_{k-1}), x \in R. \quad (29)$$

A variational inequalities form of (24)–(29) for American Option but with different initial and terminal conditions can be found in [1]. In the next section, we apply the multi-stage compound-option model (MCO) to approximate the American-style CB and calculate the constructed CB price step by step with explicit–implicit difference scheme.

4. Discretization scheme

Our goal here is to solve the free boundary problem (24)–(29) numerically. For a convertible bond on k th interval, we consider M equal sub-intervals in τ -direction. To avoid dealing with infinite value of x , we choose appropriate x_{\max} sufficiently large but still finite that the CB holder choose to convert and x_{\min} small enough that the CB holder choose opponent. We divide x -direction into N equal sub-intervals, and obtain the following mesh on $[x_{\min}, x_{\max}] \times [0, T_k - T_{k-1}]$:

$$D_k = \{(x_i, \tau_j) \in R^+ \times R^+ | x_i = (i - 1)\Delta x, i = \overline{1, N + 1}, \tau_j = (j - 1)\Delta \tau, \\ j = \overline{1, M + 1}, \Delta x = (x_{\max} - x_{\min})/N, \Delta \tau = (T_k - T_{k-1})/M\}. \quad (30)$$

A standard American-style CB can be converted into the underlying stocks anytime before and on the expiry date. By limiting the early conversion privilege to occur only at a set of predetermined instants, a CB can be modeled as a multi-stage compound-option (MCO). Without losing generality, we assume all the discrete dates $\{\tau_j | j = 1, 2, \dots, M + 1\}$ are the decision-making instants. Obviously, the value of the constructed CB converges

³ On each coupon delivery date, when the CB holder choose to convert or not, he will compare the conversion value against the coupon rate plus with the CB value at the starting point for the next new stage.

⁴ Since $V_k(x, \tau) = IS_0e^x = U_k(S, t) = IS$ when $x \in (c_k(\tau), +\infty)$. Substitute it into formula (17) and combined formula (A.5) we have $\mathcal{L}V_k(x, \tau) - rV_k(x, \tau) = \mathcal{L}^S U_k(S, t) - rU_k(S, t) = -qIS$, and obviously $\frac{\partial V_k(x, \tau)}{\partial \tau} = 0$. So we can deduce formula (28) easily.

to that of the American-style CB as $N \rightarrow \infty$, and the holder can actually carry out the conversion right at any instant prior to the expiry. Theoretically, without numerical errors, the MCO approach can give an approximation to the value of CB with a conversion of American-style with any accuracy when N is sufficiently large. In simplicity, we write $V_{i,j}$ in short for the discrete values of $V(x_i, \tau_j)$ on D_k , then $V_{i,j}$ satisfies following PIDE

$$\frac{\partial V}{\partial \tau} = \mathcal{L}V - rV, \quad x \in [x_{\min}, x_{\max}] \quad (31)$$

with boundary conditions⁵:

$$V(x_{\min}, \tau_j) = e^{-r\Delta\tau} V(x_{\min}, \tau_{j-1}) \quad (32)$$

$$V(x_{\max}, \tau_j) = IS_0 \exp(x_{\max}). \quad (33)$$

We discrete the PIDE by Explicit–Implicit difference method (EXIM). The idea of the method is to consider one part of the integral term implicitly and the remaining explicitly. The implicit part will make sure the scheme requires only a few time-steps. [12] proves that the following explicit–implicit time stepping scheme have consistency, stability and convergence properties, which treats the integral part in an explicit time stepping in order to avoid the inversion of the dense matrix J :

$$\frac{V^{j+1} - V^j}{\Delta\tau} = DV^{j+1} + JV^j \quad (34)$$

where

$$DV = (r - q + \omega)V_x - rV \quad (35)$$

$$JV = \int_{-\infty}^{\infty} [V(x + y, \tau) - V(x, \tau)]k(y)dy. \quad (36)$$

There are various approaches to evaluating the integral part $JV_{i,j}$. In this paper, we apply the method of [12] and obtain the approximation in our convertible bond case as shown in [Appendix B](#), where

$$JV_{i,j} = A(i, :) \times V(:, \tau_j) + B_i \quad \text{for } i = 2 : N. \quad (37)$$

Using the first order finite difference approximation for V_τ and central difference for V_x , we obtain the following EXIM form equation at point (x_i, τ_{j+1})

$$\begin{aligned} \frac{V_{i,j+1} - V_{i,j}}{\Delta\tau} &= (r - q + \omega) \frac{V_{i+1,j+1} - V_{i-1,j+1}}{2\Delta x} \\ &- rV_{i,j+1} + A(i, :)V(:, \tau_j) + B_i \quad \text{for } i = 2 : N. \end{aligned} \quad (38)$$

Thus,

$$\begin{aligned} &\frac{(r - q + \omega)\Delta\tau}{2\Delta x} V_{i-1,j+1} + (1 + r\Delta\tau)V_{i,j+1} - \frac{(r - q + \omega)\Delta\tau}{2\Delta x} V_{i+1,j+1} \\ &= V_{i,j} + \Delta\tau A(i, :) \times V(:, \tau_j) + \Delta\tau B_i \quad \text{for } i = 2 : N \end{aligned} \quad (39)$$

For multiple x -direction, we can simply write it as,

$$V_{2:N,j+1} = C^{-1} (V_{2:N,j} + \Delta\tau A_{\{2:N,: \}} \times V_{:,j} + \Delta\tau B_{2:N}) = C^{-1}G \quad (40)$$

where

$$G = V_{2:N,j} + \Delta\tau A_{\{2:N,: \}} \times V_{:,j} + \Delta\tau B_{2:N}$$

⁵ Note that the assumption is that the discrete value $V_{i,j}$ at time τ_j are known and we are solving for $V_{i,j+1}$ at time τ_{j+1} .

and

$$C = \begin{pmatrix} b & -a & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & a & b & -a & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a & b & -a & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & a & \dots & 0 & 0 & 0 & \dots & a & b & -a \\ 0 & 0 & 0 & a & \dots & 0 & 0 & 0 & \dots & 0 & a & b \end{pmatrix}_{(N-1) \times (N-1)} \quad (41)$$

$$a = \frac{(r - q + \omega)\Delta\tau}{2\Delta x}, \quad b = 1 + r\Delta\tau. \quad (42)$$

Of course, the boundary condition of formula (32) and (33) should also be considered, they will influence the first and last elements of G in formula (40). So in simulation, we should add the first and last elements of G by $-a * V_k(x_{min}, \tau_{j+1})$ and $a * V_k(x_{max}, \tau_{j+1})$ respectively.

By comparison with conversion value $IS_0 e^{x_i}$, we obtain values for all $V_{i,j+1}$ and optimal asset price at which the CB should be converted at time τ_{j+1} .

All in all, we summarize our numerical algorithm as follows:

Step 1: Backward calculation. For each $k = K, K - 1, \dots, 1$

- (1) Set $j = 1$ (that is $\tau_j = 0$), calculate $V_k(x, \tau_j) = H_k(x)$ for every x .
- (2) For each $j = 1, 2, 3, \dots, M$
 - (a) Calculate the boundary conditions: $V_k(x_{min}, \tau_{j+1}) = e^{-r\Delta\tau} V_k(x_{min}, \tau_j)$ and $V_k(x_{max}, \tau_{j+1}) = IS_0 \exp(x_{max})$;
 - (b) Calculate $G = V_k(x_{2:N}, \tau_j) + \Delta\tau A_{\{2:N,: \}} \times V_k(x_{1:N+1}, \tau_j) + \Delta\tau B_{2:N}$ and add the first line and the last line of column vector G by $-a * V_k(x_{min}, \tau_{j+1})$ and $a * V_k(x_{max}, \tau_{j+1})$ respectively;
 - (c) Calculate $V_k(x_{2:N}, \tau_{j+1}) = C^{-1}G$;
 - (d) Set $V_k(x_{2:N}, \tau_{j+1}) = \max(V_k(x_{2:N}, \tau_{j+1}), IS_0 e^{x_{2:N}})$.

Step 2: Choose $V_1(x = \ln(S_0), \tau = T_1)$ as the initial value of the CB price. Choose the smallest x when $V_k(x, \tau_j) = IS_0 \exp(x)$ as the optimal logarithmic asset value at every time step τ_j .

5. Numerical example

The example with application of our new model is a 5-year CB with coupon payments at the end of each year. We assume the par $C = 100$, constant conversion price $K = 10$ and the coupon rate are set to be $r_1 = 2\%$, $r_2 = 2.5\%$, $r_3 = 3\%$, $r_4 = 3.5\%$, $r_5 = 4\%$. Furthermore, let us also assume that the risk-free interest rate remains a constant for 5 years at 5%, the dynamics of underlying stock price satisfies an exponential VG process as Eq. (12) with initial price $S_0 = 10$, dividend rate $q = 0.01$ and VG parameters $\sigma = 0.1787$, $\nu = 0.1332$, $\theta = -0.3065$.⁶

We first show some convergence results of our discretization scheme. In Figs. 1 and 2, we show the results when separating the τ -direction and x -direction respectively. As shown in Fig. 1, the CB price moves upward when we fix $N = 4000$ and let the M increase from 800 to 1000, 2000. This is in accordance with the MCO model. Differently, the CB price shown in Fig. 2 moves downward when we fix $M = 1000$ and let the interval number of x range from 1000 to 2000, 4000. It is because from any line in Fig. 1 or Fig. 2 we can conclude that the CB price is a concave function on the underlying stock price, so the EXIM methods (linear interpretation in J operator approximation) will overestimate the CB price. When we fix the time interval, and increase N , it will make the CB price move downwards to the true value.

⁶ In real world, we can estimate all VG parameters with Moment Estimation method same as J. Yu et al. (2009) [25].

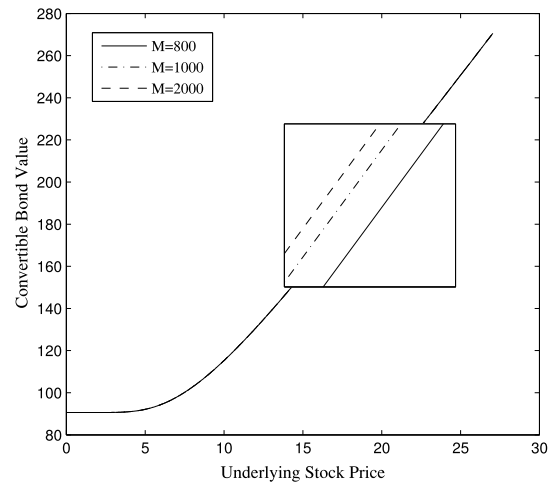


Fig. 1. Price of CB under EVG model for $N = 4000$.

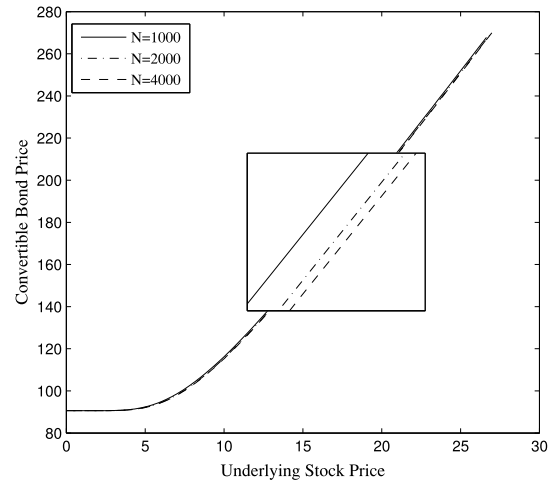


Fig. 2. Price of CB under EVG model for $M = 1000$.

Figs. 3 and 4 show the CB price and optimal conversion price (OCP) respectively when we increase both of the N and M .⁷ There are two parts in Fig. 4, the left part shows the whole picture of OCP, and the right part zooms the left part when the upper limit of y-axis (that is the largest stock price) is set to be equal to 150. You will find Figs. 5 and 8 are both zoomed similar as Fig. 4. We can find that both of the CB prices in Fig. 3 and optimal conversion price in Fig. 4 convergent quickly. As shown in Table 1, the calculation time for different discretization is $O(MN^2)$ (except the first line⁸), which has been proved in [9].

For comparison purpose, we also price the convertible bond under Geometric Brownian Motion (GBM) assumption. To ensure the same statistical expectation and variance of the log returns of underlying stock price for both EVG and GBM models, we assume $S_t = S_0 \exp((\mu - q)t + \sigma_w W(t))$ for GBM approach in numerical calculation, where $\mu = m + \theta$ and $\sigma_w = \sqrt{\theta^2 v + \sigma^2}$.

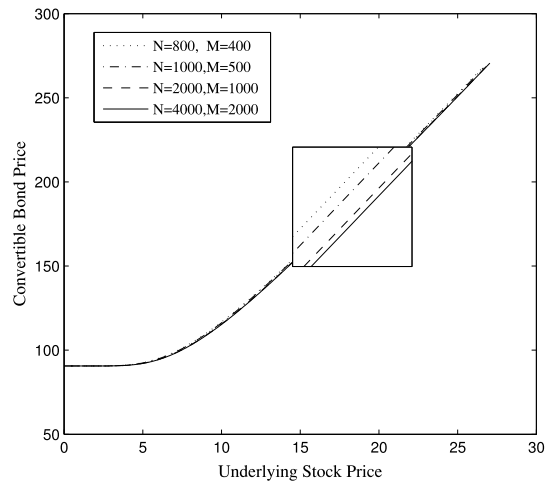
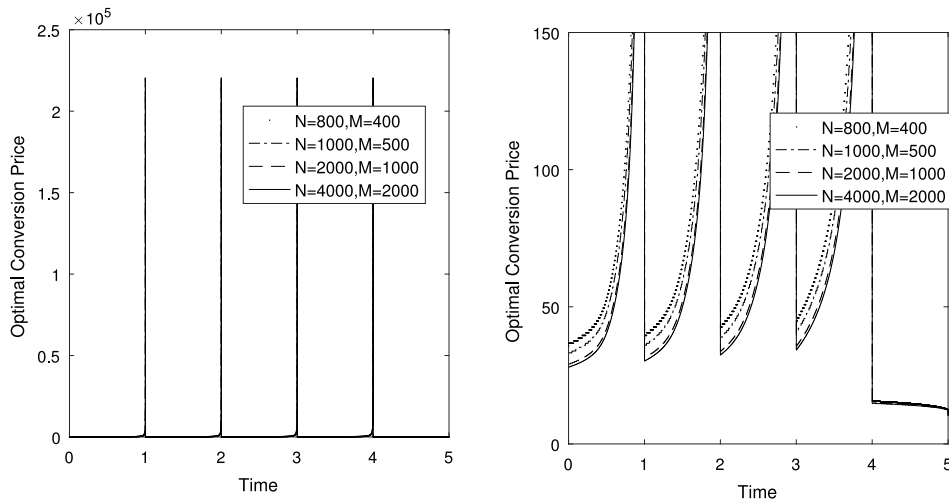
⁷ For reasons of space, only a ratio of $N : M = 2 : 1$ is shown.

⁸ There exists a fix time consumption for every simulation, it will influence much more when the total simulation time is small relatively. So time consumption rule of $O(MN^2)$ is not obviously when N and M are relatively small.

Table 1

Calculation time for CB pricing with different discretization in case of $q = 0.01$ (Pentium IV, 2.66 GHz).

N	M	Time (s)
400	200	2.001871
800	400	8.373386
1000	500	14.862343
2000	1000	92.516224
4000	2000	640.736400

**Fig. 3.** Price of CB under EVG model for different discretization numbers.**Fig. 4.** OCP of CB under EVG model for different discretization numbers.

Based on GBM model, the convertible bond value satisfies following PDE, which can be calculated similarly by EXIM difference method.

$$\frac{\partial V}{\partial \tau} = \frac{\sigma_w^2}{2} \frac{\partial^2 V}{\partial x^2} + (r - q) \frac{\partial V}{\partial x} - rV. \quad (43)$$

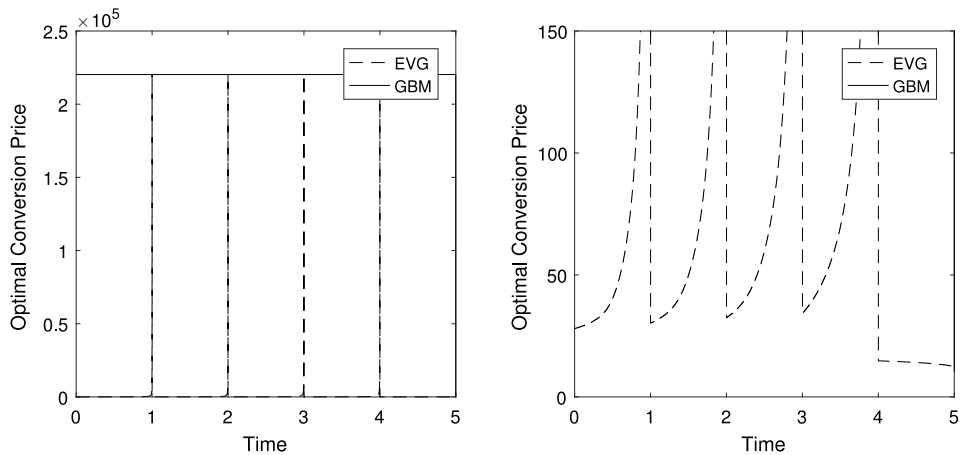


Fig. 5. Comparison of OCP under both EVG and GBM models ($q = 0.01$, $N = 4000$, $M = 2000$).

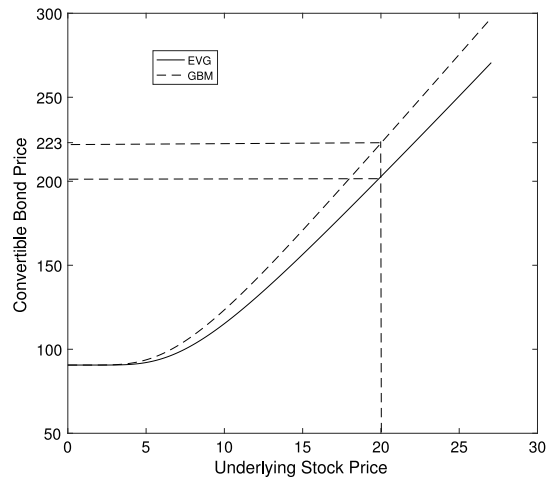


Fig. 6. Comparison of CB price under both EVG and GBM models ($q = 0.01$, $N = 4000$, $M = 2000$).

Figs. 5 and 6 respectively show the OCP and CB price for both GBM model and EVG model with parameters $q = 0.01$, $N = 4000$, $M = 2000$. From the left part in Fig. 5, we can find that the OCP in GBM model is a constant equal to $10 \times e^{10} \approx 2.2026 \times 10^5$, which is the maximum boundary of the stock price in simulation.⁹ It means that the CB holder will never execute the conversion option until the maturity in GBM model. However, from the right part in Fig. 5,¹⁰ We can find it is possible for the CB holder to convert it before the maturity in EVG model. One should also observe that when the time is closer to coupon delivery date, the OCP shoot the sky rapidly in EVG case, which is coincident with the result of [2].

From Fig. 6, we find that the CB price in GBM model is higher than that in EVG case especially when the underlying stock price becomes large. The differences can be substantial, for example, when the stock price is about 20.04, the CB price under the EVG model is about 10% lower than that under the GBM model.

Table 2 compares the CB price based on the two models in different q and initial stock prices. It shows that the difference between these two models increase when the initial stock price increases and decrease when the dividend rate q increases.

⁹ In simulation, we set the upper boundary of the stock yield return as 10 and set the initial stock price equal to 10.

¹⁰ Because the OCP in GMB model is out of the scope of the y-axis, so there are no real line to indicate the OCP under GBM model.

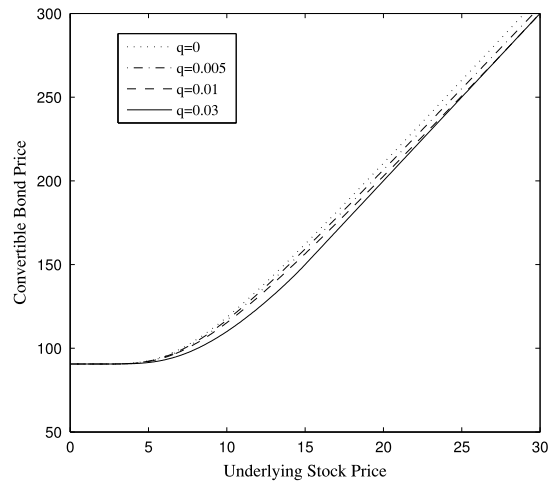


Fig. 7. Comparison of CB price under EVG model with different q .

Table 2

CB price comparison between EVG model and GBM model with different dividend rates and different discretization when $N = 4000$, $M = 2000$ (Pentium IV, 2.66 GHz).

q	Model	$S_0 = 4.99$	$S_0 = 10$	$S_0 = 20.04$	$S_0 = 39.95$	$S_0 = 80.04$
0	EVG	92.53	118.53	210.93	409.41	810.64
	GBM	94.41	127.86	233.88	456.06	904.17
0.005	EVG	92.30	116.83	206.92	401.51	800.45
	GBM	94.03	125.62	228.40	445.04	882.08
0.01	EVG	92.09	115.26	203.13	399.48	800.45
	GBM	93.67	123.47	223.07	434.29	860.54
0.03	EVG	91.47	109.98	200.37	399.48	800.45
	GBM	92.57	116.14	204.52	399.48	800.45

At last, we give the numerical results on sensitivity of parameter q . As shown in Fig. 7, when the dividend rate varies from 0, 0.005, 0.01 to 0.03, the CB price decreases because we assume the consistent convert price.¹¹ We also observe that the optimal conversion price in Fig. 8 is higher when the dividend rate is lower and it approaches to infinity when $q = 0$, which is coincident with the American option case.

6. Conclusion and future work

We apply the Exponential Variance Gamma process into Convertible Bond pricing model. For numerical calculation, we develop a discrete scheme, which combines MCO model and EXIM difference method.

A comparison revealed that the convertible bond prices under EVG model are lower than those under GBM model, especially when the initial stock price is higher. That is because the EVG model can capture the skewness and leptokurtosis features of the yield of stock price. The optimal conversion prices under the two models are significantly different. The convertible bond will never execute the convert option till the end of the maturity. But it is possible for the CB holder to convert it before the maturity in EVG model. We are led to conclude that even though the pricing of CB under EVG model is more complicated than the previous GBM-based models, it is important from the perspective of correctly accounting for the values of these instruments.

The methods developed here would be applicable to a wide class of convertible bonds pricing such as those with call or put provisions, and reset clause etc. In future study, we should build more realistic models that can capture many more features of convertible bond in reality.

¹¹ If the conversion price changes with dividend rate, then it is same with the case $q = 0$.

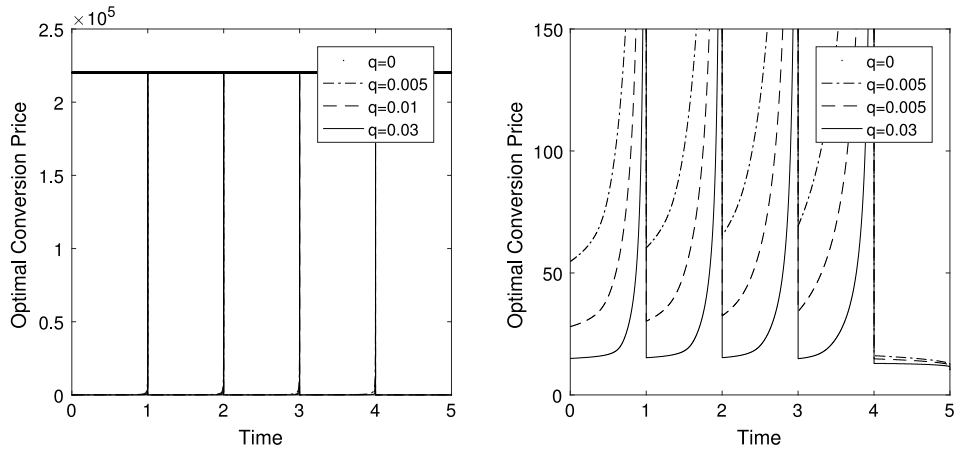


Fig. 8. Comparison of CB OCP under EVG model with different q .

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Competing interests statement

The authors declare that they have no competing financial interests.

Appendix A. Proof of γ'

Assume the Lévy triplet of $X_t := (r - q + \omega)t + X_t^{VG}$ is $[\gamma', 0, \nu_{VG}(dx)]$, thus we have its characteristic exponent:

$$\phi_X(u) = i\gamma'u + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux1_{\{|x|<1\}})k(x)dx. \quad (\text{A.1})$$

By no-arbitrage of S_t , $E(e^{-(r-q)t}S_t) = S_0$, equivalently, $E(e^{X_t}) = e^{(r-q)t}$, thus we obtain $r - q = \phi_X(-i)$, that is

$$\gamma' + \int_{-\infty}^{\infty} (e^x - 1 - x1_{\{|x|<1\}})k(x)dx = r - q, \quad (\text{A.2})$$

equivalently

$$\gamma' = r - q - \int_{-\infty}^{\infty} (e^x - 1 - x1_{\{|x|<1\}})k(x)dx. \quad (\text{A.3})$$

According to (A.5), we finish the proof thus obtain

$$\gamma' = r - q + \omega + \int_{-\infty}^{\infty} x1_{\{|x|<1\}}k(x)dx \quad (\text{A.4})$$

$$\begin{aligned} \int_{-\infty}^{\infty} (e^y - 1)k(y)dy &= \int_0^{\infty} (e^{-y} - 1)k(-y)dy + \int_0^{\infty} (e^y - 1)k(y)dy \\ &= \lim_{\varepsilon \rightarrow 0} \left(\int_{\varepsilon}^{\infty} (e^{-y} - 1) \frac{e^{-\lambda_n y}}{vy} dy + \int_{\varepsilon}^{\infty} (e^y - 1) \frac{e^{-\lambda_p y}}{vy} dy \right) \\ &= \lim_{\varepsilon \rightarrow 0} \left(\int_{\varepsilon}^{\infty} \frac{e^{-(\lambda_n+1)y}}{vy} dy - \int_{\varepsilon}^{\infty} \frac{e^{-\lambda_n y}}{vy} dy + \int_{\varepsilon}^{\infty} \frac{e^{-(\lambda_p-1)y}}{vy} dy - \int_{\varepsilon}^{\infty} \frac{e^{-\lambda_p y}}{vy} dy \right) \end{aligned}$$

$$\begin{aligned}
&= \lim_{\varepsilon \rightarrow 0} \left(\frac{1}{v} E((\lambda_n + 1)\varepsilon) - \frac{1}{v} E(\lambda_n \varepsilon) + \frac{1}{v} E((\lambda_p - 1)\varepsilon) - \frac{1}{v} E(\lambda_p \varepsilon) \right) \\
&= \lim_{\varepsilon \rightarrow 0} \left(-\frac{1}{v} \int_{\lambda_n}^{\lambda_n+1} \frac{e^{-\varepsilon y}}{vy} dy + -\frac{1}{v} \int_{\lambda_p-1}^{\lambda_p} \frac{e^{-\varepsilon y}}{vy} dy \right) \\
&= -\frac{1}{v} \ln\left(\frac{\lambda_n + 1}{\lambda_n}\right) + \frac{1}{v} \ln\left(\frac{\lambda_p}{\lambda_p - 1}\right) = \frac{1}{v} \ln\left(\frac{\lambda_n \lambda_p}{(\lambda_p - 1)(\lambda_n + 1)}\right) \\
&= \frac{1}{v} \ln\left(\frac{2/(\sigma^2 v)}{2/(\sigma^2 v) - 2\theta/\sigma^2 - 1}\right) = -\frac{1}{v} \ln(1 - \theta v - \sigma^2 v/2) = -\omega.
\end{aligned} \tag{A.5}$$

Appendix B. Approximation of J

As Lévy measure $k(y)$ is singular at $y = 0$, we divide the domain of integration into six sub-intervals.

$$\begin{aligned}
&\int_{-\infty}^{\infty} [V(x_i + y, \tau_j) - V_{i,j}]k(y)dy = \int_{-\infty}^{x_1-x_i} [V(x_i + y, \tau_j) - V_{i,j}]k(y)dy \\
&+ \int_{x_1-x_i}^{-\Delta x} [V(x_i + y, \tau_j) - V_{i,j}]k(y)dy + \int_{-\Delta x}^0 [V(x_i + y, \tau_j) - V_{i,j}]k(y)dy \\
&+ \int_0^{\Delta x} [V(x_i + y, \tau_j) - V_{i,j}]k(y)dy + \int_{\Delta x}^{x_{N+1}-x_i} [V(x_i + y, \tau_j) - V_{i,j}]k(y)dy \\
&+ \int_{x_{N+1}-x_i}^{\infty} [V(x_i + y, \tau_j) - V_{i,j}]k(y)dy = I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned} \tag{B.1}$$

Notice that, for some special value of i ($i = 1, 2, N, N + 1$), some of these sub-integral parts equal to zero, thus we have

$$JV = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \dots & & & \dots & & \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}_{(N+1) \times 6} \times \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{pmatrix} \tag{B.2}$$

B.1. Approximation of I_1

For $y \in (-\infty, x_1 - x_i)$, we have $x_i + y < x_1 = x_{\min}$, thus $V(x_i + y, \tau_j) = V_{1,j}$,

$$\begin{aligned}
I_1 &= \int_{-\infty}^{x_1-x_i} (V_{1,j} - V_{i,j})k(y)dy = (V_{1,j} - V_{i,j}) \int_{(i-1)\Delta x}^{\infty} \frac{e^{-\lambda_n y}}{vy} dy \\
&= \frac{1}{v} E((i-1)\Delta x \lambda_n) V_{1,j} - \frac{1}{v} E((i-1)\Delta x \lambda_n) V_{i,j}
\end{aligned} \tag{B.3}$$

where, for any $x > 0$, $E(x) = \int_x^{\infty} \frac{e^{-y}}{y} dy$. Specifically, if $i = 1$, $I_1=0$.

B.2. Approximation of I_2

$$I_2 = \int_{\Delta x}^{(i-1)\Delta x} [V(x_i - y, \tau_j) - V_{i,j}]k(-y)dy = \sum_{k=1}^{i-2} \int_{k\Delta x}^{(k+1)\Delta x} [V(x_i - y, \tau_j) - V_{i,j}] \frac{e^{-\lambda_n y}}{vy} dy$$

For $y \in (k\Delta x, (k+1)\Delta x)$, we have

$$V(x_i - y, \tau_j) \approx \frac{V(x_i - (k+1)\Delta x, \tau_j) - V(x_i - k\Delta x, \tau_j)}{\Delta x}(y - k\Delta x) + V(x_i - k\Delta x, \tau_j) = \frac{V_{i-k-1,j} - V_{i-k,j}}{\Delta x}(y - k\Delta x) + V_{i-k,j}$$

and

$$\begin{aligned} I_2 &= \sum_{k=1}^{i-2} \int_{k\Delta x}^{(k+1)\Delta x} [V_{i-k,j} + \frac{V_{i-k-1,j} - V_{i-k,j}}{\Delta x}(y - k\Delta x) - V_{i,j}] \frac{e^{-\lambda_n y}}{vy} dy \\ &= \sum_{k=1}^{i-2} \int_{k\Delta x}^{(k+1)\Delta x} [(1+k)V_{i-k,j} - kV_{i-k-1,j} - V_{i,j}] \frac{e^{-\lambda_n y}}{vy} dy \\ &\quad + \sum_{k=1}^{i-2} \int_{k\Delta x}^{(k+1)\Delta x} \frac{V_{i-k-1,j} - V_{i-k,j}}{\Delta x} \frac{1}{v} e^{-\lambda_n y} dy \\ &= \sum_{k=1}^{i-2} \frac{1}{v} [(1+k)V_{i-k,j} - kV_{i-k-1,j} - V_{i,j}] (E(k\Delta x \lambda_n) - E((k+1)\Delta x \lambda_n)) \\ &\quad + \sum_{k=1}^{i-2} \frac{V_{i-k-1,j} - V_{i-k,j}}{v\Delta x \lambda_n} (e^{-\lambda_n k\Delta x} - e^{-\lambda_n (k+1)\Delta x}) \\ &= \sum_{k=1}^{i-2} (\frac{1+k}{v} (E(k\Delta x \lambda_n) - E((k+1)\Delta x \lambda_n)) - \frac{1}{v\Delta x \lambda_n} (e^{-\lambda_n k\Delta x} - e^{-\lambda_n (k+1)\Delta x})) V_{i-k,j} \\ &\quad - \sum_{k=1}^{i-2} \frac{1}{v} (E(k\Delta x \lambda_n) - E((k+1)\Delta x \lambda_n)) V_{i,j} \\ &\quad + \sum_{k=1}^{i-2} (-\frac{k}{v} (E(k\Delta x \lambda_n) - E((k+1)\Delta x \lambda_n)) + \frac{1}{v\Delta x \lambda_n} (e^{-\lambda_n k\Delta x} - e^{-\lambda_n (k+1)\Delta x})) V_{i-k-1,j}. \end{aligned}$$

B.3. Approximation of I_3, I_4

For $y \in (0, \Delta x)$, we have $V(x_i - y, \tau_j) - V(x_i, \tau_j) \approx (V_{i-1,j} - V_{i,j})/\Delta x$, and $V(x_i + y, \tau_j) - V(x_i, \tau_j) \approx (V_{i+1,j} - V_{i,j})/\Delta x$, thus

$$I_3 = \int_0^{\Delta x} \frac{V_{i-1,j} - V_{i,j}}{\Delta x} \frac{e^{-\lambda_n y}}{v} dy = \frac{1}{v\Delta x \lambda_n} (1 - e^{-\lambda_n \Delta x}) (V_{i-1,j} - V_{i,j})$$

$$I_4 = \int_0^{\Delta x} \frac{V_{i+1,j} - V_{i,j}}{\Delta x} \frac{e^{-\lambda_p y}}{v} dy = \frac{1}{v\Delta x \lambda_p} (1 - e^{-\lambda_p \Delta x}) (V_{i+1,j} - V_{i,j}).$$

B.4. Approximation of I_5

$$I_5 = \int_{\Delta x}^{(N-i+1)\Delta x} [V(x_i + y, \tau_j) - V_{i,j}] k(y) dy = \sum_{k=1}^{N-i} \int_{k\Delta x}^{(k+1)\Delta x} [V(x_i + y, \tau_j) - V_{i,j}] \frac{e^{-\lambda_p y}}{vy} dy$$

For $y \in (k\Delta x, (k+1)\Delta x)$, we have

$$V(x_i + y, \tau_j) \approx \frac{V(x_i + (k+1)\Delta x, \tau_j) - V(x_i + k\Delta x, \tau_j)}{\Delta x}(y - k\Delta x) + V(x_i + k\Delta x, \tau_j) = V_{i+k,j} + \frac{V_{i+k+1,j} - V_{i+k,j}}{\Delta x}(y - k\Delta x)$$

$$\begin{aligned}
I_5 &= \sum_{k=1}^{N-i} \int_{k\Delta x}^{(k+1)\Delta x} [V_{i+k,j} + \frac{V_{i+k+1,j} - V_{i+k,j}}{\Delta x} (y - k\Delta x) - V_{i,j}] \frac{e^{-\lambda_p y}}{vy} dy \\
&= \sum_{k=1}^{N-i} \int_{k\Delta x}^{(k+1)\Delta x} [(1+k)V_{i+k,j} - kV_{i+k+1,j} - V_{i,j}] \frac{e^{-\lambda_p y}}{vy} dy \\
&\quad + \sum_{k=1}^{N-i} \int_{k\Delta x}^{(k+1)\Delta x} \frac{V_{i+k+1,j} - V_{i+k,j}}{v\Delta x} \frac{e^{-\lambda_p y}}{v} dy \\
&= \sum_{k=1}^{N-i} \frac{1}{v} [(1+k)V_{i+k,j} - kV_{i+k+1,j} - V_{i,j}] (E(k\Delta x\lambda_p) - E((k+1)\Delta x\lambda_p)) \\
&\quad + \sum_{k=1}^{N-i} \frac{V_{i+k+1,j} - V_{i+k,j}}{v\Delta x\lambda_p} (e^{-\lambda_p k\Delta x} - e^{-\lambda_p (k+1)\Delta x}) \\
&= \sum_{k=1}^{N-i} \left(\frac{1}{v} (1+k)(E(k\Delta x\lambda_p) - E((k+1)\Delta x\lambda_p)) - \frac{1}{v\Delta x\lambda_p} (e^{-\lambda_p k\Delta x} - e^{-\lambda_p (k+1)\Delta x}) \right) V_{i+k,j} \\
&\quad - \sum_{k=1}^{N-i} \frac{1}{v} (E(k\Delta x\lambda_p) - E((k+1)\Delta x\lambda_p)) V_{i,j} \\
&\quad + \sum_{k=1}^{N-i} \left(-\frac{k}{v} (E(k\Delta x\lambda_p) - E((k+1)\Delta x\lambda_p)) + \frac{1}{v\Delta x\lambda_p} (e^{-\lambda_p k\Delta x} - e^{-\lambda_p (k+1)\Delta x}) \right) V_{i+k+1,j}.
\end{aligned}$$

B.5. Approximation of I_6

For $y \in ((N+1-i)\Delta x, \infty)$, we have $V(x_i + y, \tau_j) = IS_0 e^{x_i + y}$, thus

$$\begin{aligned}
I_6 &= IS_0 e^{x_i} \int_{(N+1-i)\Delta x}^{\infty} \frac{e^{-(\lambda_p - 1)y}}{vy} dy - V_{i,j} \int_{(N+1-i)\Delta x}^{\infty} \frac{e^{-\lambda_p y}}{vy} dy \\
&= -\frac{1}{v} E((N+1-i)\lambda_p \Delta x) V_{i,j} + \frac{1}{v} IS_0 e^{x_i} E((N+1-i)(\lambda_p - 1)\Delta x).
\end{aligned}$$

Specifically, if $i = N+1$,

$$\begin{aligned}
I_6 &= \int_0^{\infty} (IS_0 e^{x_i + y} - IS_0 e^{x_i}) dy \\
&= IS_0 e^{x_{N+1}} \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} (e^y - 1) dy = IS_0 e^{x_{N+1}} \frac{1}{v} \log \frac{\lambda_p}{\lambda_p - 1}.
\end{aligned}$$

B.6. Approximation of $JV_{i,j}$

Notice that $I_1 - I_6$ are linear functions of $V(\cdot, j)$ with coefficients which independent to j but determined by the discretization of x -direction and Lévy triplet of VG process. Thus we denote the same coefficient matrix $A := A_k (k = 1, \dots, K)$ for each conversion stage and obtain the approximation form of integral part as

$$JV_k(\cdot, j) = A \times V_k(\cdot, j) + B, \quad (\text{B.4})$$

where A can be easily obtained by a straightforward algebraic calculations from (B.2) and approximations of the six sub-integrals, and

$$B_i = \begin{cases} IS_0 e^{x_i} E((N+1-i)(\lambda_p - 1)\Delta x)/v & \text{if } i < N+1 \\ IS_0 e^{x_i} \frac{1}{v} \log \frac{\lambda_p}{\lambda_p - 1} & \text{if } i = N+1. \end{cases}$$

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